# Absorption and emission

# GOALS :

Understand the width and profiles of the spectral lines

Link them to

- excitation and relaxation of atoms
- population of the different atomic energy levels
- gas composition

We will go through a microscopic and quantic description

In order to fully apreciate its need and power, we will start by a classical approach and show its limits.

let's express $\kappa$ as a function of physical quantities

To this end, we will use the equations which describe the propagation of waves, even if refracted and their absorption.

$\rightarrow$ Maxwell equations

We use the convention
$$E_\nu = E_\nu^0 \, e^{i(\omega t - kx)}$$
$$= E_\nu^0 \, e^{i 2\pi \nu \left(t - \frac{\hat{n}}{c} x\right)}$$

$$\boxed{\hat{n} = n - i n' \qquad \begin{array}{l} n = \text{refraction index} \\ n' = \text{absorption index} \end{array}}$$

if $\hat{n}$ is complex : $E_\nu = E_\nu^0 \, e^{i 2\pi \nu \left(t - \frac{n}{c} x + i n' x\right)}$

We can isolate the real and complex (imaginary) terms :
$$E_\nu = E_\nu^0 \, e^{-\frac{n' 2\pi \nu}{c} x} \, e^{2\pi i \nu \left(t - \frac{n x}{c}\right)}$$

Phase velocity $= \dfrac{c}{n}$

We have a damped wave
$$I_\nu = I_\nu^0 \, e^{-\frac{4 n' \pi \nu}{c} x}$$
$$= I_\nu^0 \, e^{-\kappa_\nu \rho x}$$

$$\Rightarrow \kappa_\nu \rho = \frac{4\pi n' \nu}{c}$$

We define $\boxed{\mathcal{K}' = K\rho}$

$$\mathcal{K}' = \frac{4\pi\nu n'}{c} = \frac{2\omega n'}{c} \quad \text{with} \quad \omega = 2\pi\nu$$

Hence we need to know $n'$ in order to calculate $\mathcal{K}'$

let's consider the motion of $e^-$ in an electromagnetic field

Equation of motion : $m \frac{d^2\vec{u}}{dt^2} = \sum \vec{F}$

$\boxed{\sum \vec{F} ?}$

$e^-$ : mass $\equiv m_e$ charge $\equiv e$

- Lorentz force $e\left(\vec{E} + \frac{d\vec{u}}{dt} \wedge \vec{B}\right)$

- Restoring force due to the interaction with the nuclei, acting to bring $e^-$ back to its original position : $-K\vec{u}$

- Resistive (friction) force proportional to the $e^-$ velocity, from the collisions with the surrounding particles
$$-m_e \gamma \frac{d\vec{u}}{dt}$$

in 1D
$$m_e \ddot{x} = -m_e \gamma \frac{dx}{dt} - Kx + e E_\omega^0 e^{i\omega t} \qquad (1)$$

$\rightarrow$ We will neglect $\vec{B}$

$\rightarrow$ As to $x$, in case $\vec{E}$ slowly varying over $\gg 1\text{Å}$, the displacement is $\sim 1\text{Å}$, so we can simplify with $x = 0$

(1) is the equation of an oscillator of solution

$$x(t) = X_0 e^{i(\omega t + \varphi)} = x_0 e^{i\omega t}$$

with $x_0 = X_0 e^{i\varphi}$

$$(1) \equiv \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{e}{m_e} E_\omega^0 e^{i\omega t} \quad \text{with } \omega_0^2 = \frac{K}{m_e}$$

we can insert $x(t)$ in the equation,

$$-\omega^2 x_0 + i\omega \gamma x_0 + \omega_0^2 x_0 = \frac{e}{m_e} E_\omega^0$$

$$\Rightarrow x_0 = \frac{e}{m_e} \frac{E_\omega^0}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

let $N$ be the number of dipoles per unit volume
let $\vec{p}$ be the dipole moment of each dipole
let $x$ be the displacement of the negative charges with regard to the barycenter of the positive charges

polarisation is $\vec{P} = N\vec{p} = Ne\vec{x} = \chi_r \vec{E}$

$$\chi_r = \frac{Ne}{E} x$$

Electric susceptibility.

$$\chi_r = \frac{Ne^2}{m_e} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$$ . Susceptibility

and

$$\varepsilon_r = 1 + 4\pi\chi_r = \hat{n}^2$$ dielectric constant

$$\varepsilon_r = 1 + \frac{4\pi Ne^2}{m_e} \frac{1}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

We are looking for $\hat{n}$ ! $\hat{n}^2 = (n - in')^2 = n^2 - n'^2 - 2inn'$

In order to isolate the real and imaginary terms, we must multiply by $(\omega_0^2 - \omega^2) + i\gamma\omega$

This gives

$$\begin{cases} n^2 - n'^2 = 1 + \dfrac{4\pi Ne^2}{m_e} \dfrac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \\ nn' = \dfrac{2\pi Ne^2}{m_e} \dfrac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \end{cases}$$

$n^2 - n'^2 \longrightarrow$ real part, refraction

$nn' \longrightarrow$ imaginary part, absorption

## Approximation 1

In low-density medium, such as the stellar atmospheres, or the interstellar medium, one can take the values of the vacuum

$$n \simeq 1 \quad n' \ll 1$$

$$\Rightarrow \quad nn' \simeq n' \quad \text{and} \quad n^2 - n'^2 \sim n^2$$

## Refraction :

$$n^2 - 1 = \frac{4\pi N e^2}{m_e} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

$n^2 - 1 = (n+1)(n-1)$ and $n \sim 1$

$n^2 - 1 \simeq 2(n-1)$

$$\Rightarrow \quad n - 1 \simeq \frac{2\pi N e^2}{m_e} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

we can also consider the relation with the frequency $\nu$

$$\omega = \frac{2\pi}{T} \qquad k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi\nu$$

$$n - 1 = \frac{N e^2}{2\pi m_e} \frac{\nu_0^2 - \nu^2}{(\nu_0^2 - \nu^2)^2 + \frac{\gamma^2 \nu^2}{4\pi^2}}$$

## absorption

$$K_\nu' = \frac{4\pi \nu n'}{c} = \frac{2\omega n'}{c}$$ was our starting point

$$nn' \sim n' \sim \frac{2\pi N e^2}{m_e} \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

$$\Rightarrow K_\nu' = \frac{\pi N e^2}{m_e} \frac{4 \gamma \nu^2}{4\pi^2 (\nu_0^2 - \nu^2)^2 + \gamma^2 \nu^2}$$

## Approximation 2

Close to resonance ( $h\nu = \Delta$ Energy)

$$\nu_0^2 - \nu^2 \simeq 2\nu(\nu_0 - \nu)$$

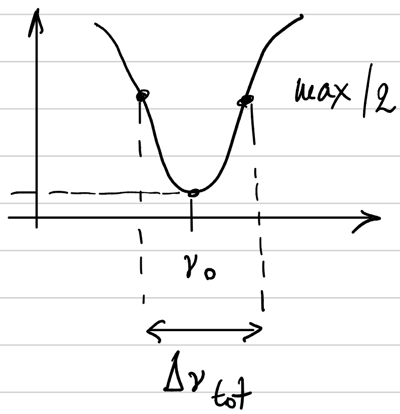
$$n - 1 \simeq \frac{N e^2}{4\pi m_e \nu} \frac{\nu_0 - \nu}{(\nu_0 - \nu)^2 + \left(\frac{\gamma}{4\pi}\right)^2}$$

$$K_\nu \simeq \frac{\pi N e^2}{m_e c} \underbrace{\frac{\gamma}{4\pi^2 (\nu_0 - \nu)^2 + \frac{\gamma^2}{4}}}_{\text{Lorentz Profile}}$$

The width of the Lorentz profile = natural width of a line

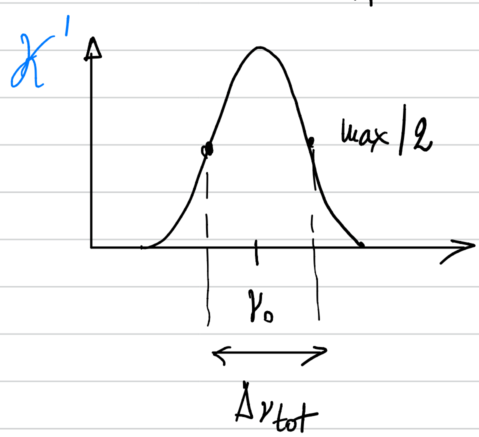
$$K' = \frac{\pi N e^2}{m_e c} \frac{\gamma}{4\pi^2 \Delta\nu + \frac{\gamma^2}{4}} \quad \text{with} \quad \Delta\nu = \nu_0 - \nu$$

Spectral line




$\equiv$

$K'$ is maximum for $\nu = \nu_0$




$$@ \max \quad K'_{max} = \frac{\pi N e^2}{m_e c} \frac{\gamma}{\frac{\gamma^2}{4}} = \frac{4\pi N e^2}{m_e c} \frac{1}{\gamma}$$

$$\frac{1}{2} K'_{max} = \frac{2\pi \cancel{N e^2}}{\cancel{m_e c}} \frac{1}{\gamma} = \frac{\cancel{\pi N e^2}}{\cancel{m_e c}} \frac{\gamma}{4\pi^2 \Delta\nu^2_{1/2} + \frac{\gamma^2}{4}}$$

$$\frac{2}{\gamma} = \frac{\gamma}{4\pi^2 \Delta\nu^2_{1/2} + \frac{\gamma^2}{4}}$$

$$\frac{2}{\gamma}\left(4\pi^2 \Delta\nu^2_{1/2} + \frac{\gamma^2}{4}\right) = \gamma$$

$$4\pi^2 \Delta\nu^2_{1/2} = \frac{\gamma^2}{2} - \frac{\gamma^2}{4} = \frac{\gamma^2}{4}$$

$$\Delta\nu^2_{1/2} = \frac{\gamma^2}{16\pi^2} \Rightarrow \Delta\nu_{1/2} = \frac{\gamma}{4\pi} \Rightarrow \Delta\nu_{tot} = \frac{\gamma}{2\pi}$$

In "practical" terms:

$\frac{1}{\gamma}$ = time for the dipole to decrease by a factor e

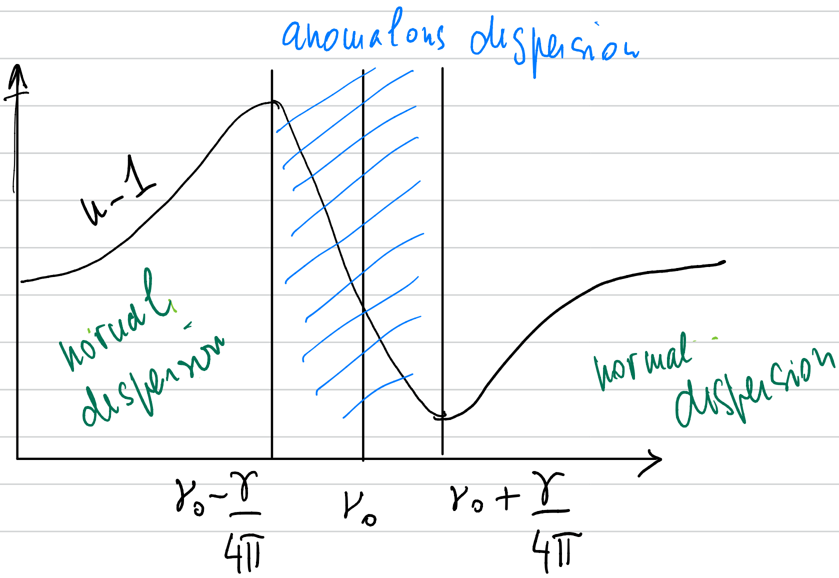
$$\gamma = \frac{2}{3} e^2 \frac{\omega_0^2}{c^3 m_e 4\pi \varepsilon_0}$$

$\varepsilon_0$ = vacuum permittivity = $8.85\ 10^{-12}\ C^2 N^{-1} m^{-2}$

lifetime of an energy level $= \frac{1}{\gamma} \simeq 10^{-8}$ s in the optical
$\sim 5000$ Å

Natural width $\Delta\lambda = \frac{4\pi}{3} \frac{e^2}{m_e c^2} = 0.12$ mÅ

HARPS $R = \frac{\lambda}{\Delta\lambda} = 10^5$ @ 5000Å $\Delta\lambda = 0.05$ Å ..... (x400)




Normality ≡ refraction ↗ with $\gamma$

↑ maximum absorption

The total opacity is obtained by integrating the Lorentz profile over all frequencies.

$$K' = \frac{\pi N e^2}{m_e c} \int_0^{\infty} \frac{\pi \gamma \, d\nu}{4\pi^2 (\nu_0 - \nu)^2 + \frac{\gamma^2}{4}}$$

let $x = 4\pi(\nu - \nu_0)/\gamma \rightarrow \frac{N e^2}{m_e c} \int_{-\infty}^{+\infty} \frac{dx}{\underbrace{x^2+1}_{\left[\arctan x\right]_{-\infty}^{+\infty} = \pi}}$

Note: $-\infty$ comes from $\nu = 0$ $x \rightarrow -\frac{4\pi \nu_0}{\gamma}$

(In our notations $\gamma = \frac{\Gamma}{m_e}$ ; $\Gamma \equiv$ Friction coefficient in kg/s

et $\Gamma = \frac{N_e e^2}{\sigma_0}$ ; $\sigma_0$ electronic conductivity $> 10^7$)

It follows $K' = \frac{\pi N e^2}{m_e c}$ $m_e = 9.109 \; 10^{-31}$ kg

$-e = -1.602 \; 10^{-19}$ C

Classical physics considers only one single opacity whatever the transition!

Quantum mechanics shows that the opacity can vary a lot depending on the transition

One usually write $\kappa' = \dfrac{\pi N_i e^2}{m_e c} f_{ij}$

$f_{ij}$ = oscillator force of the transition $i \leftrightarrow j$

$N_i$ = number of dipoles / oscillators per unit volume which can absorb one photon for the transition $ij$

$f_{ij} \longrightarrow 1$ for the strong transitions / lines (strong opacity)

$f(H_\alpha) = 0.641$ $\quad f(H_\beta) = 0.119$

In quantum mechanics $\Delta E = \dfrac{\hbar}{T}$ uncertainty on the atom energy that spend a time $T$ in the state of energy $E_0$

Its energy lies between $E_0 - \dfrac{\hbar}{T}$ and $E_0 + \dfrac{\hbar}{T}$

This produces the width of the line

Let's consider 2 levels : lower $= i$ upper $= s$.

$\gamma \longrightarrow \Gamma_i + \Gamma_s$ Sum of dampings of the two levels

$N \longrightarrow N_i f_{is}$ $f_{is}$ = oscillator force = number of $e^-$ in atom that can make the transition $i \to s$ as calculated in laboratories.

$N_i$ = number of $e^-$ on the energy level $i$ as given by the Boltzman – Saha equation

$$\nu_0 \longrightarrow \nu_{is} = \frac{E_s - E_i}{h}$$

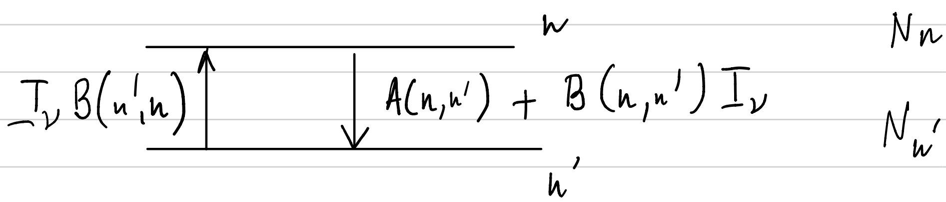
It follows

$$K_\nu^r = \frac{\pi e^2}{m_e c} N_i f_{is} \frac{\Gamma_i + \Gamma_s}{4\pi(\nu_0 - \nu)^2 + \left(\frac{\Gamma_i + \Gamma_s}{4}\right)^2}$$

## Absorption, spontaneous emission, stimulated emission

Einstein coefficients

Given 2 quantized energy levels $n'$ and $n$




$A(n,n')$ = Spontaneous emission, does not need incident radiation
$B(n,n')$ = stimulated emission, transition rate
$B(n',n)$ = absorption, transition rate

In the case of a black body, the thermal equilibrium requires that the number of emission = nb of absorptions

$$N_{n'} B_{n'n} I_\nu = N_n \left( A_{nn'} + B_{nn'} I_\nu \right)$$

leading to

$$I_\nu = \frac{N_n A_{nn'}}{\left( N_{n'} B_{n'n} - N_n B_{nn'} \right)}$$

$$I_\nu = \frac{A_{nn'}}{B_{n'n}} \frac{1}{\frac{N_{n'}}{N_n} - \frac{B_{nn'}}{B_{n'n}}}$$

For an ideal gas, the population of the energy levels follows the Maxwell - Boltzmann law

nb of particles

$$N_i = \frac{N}{\sum_j g_j e^{-E_j/kT}} \, g_i \, \underbrace{e^{-E_i/kT}}_{\text{probability to find a particle on the level } i}$$

Temperature of the system

degeneracy of state i

One can write $\quad \dfrac{N_n}{N_{n'}} = \dfrac{g_n}{g_{n'}} e^{-\frac{E_n - E_{n'}}{kT}} = \dfrac{g_n}{g_{n'}} e^{-\frac{h\nu}{kT}}$

which we can insert in the formula of $I_\nu$

$$I_\nu = \frac{A_{nn'}}{B_{n'n}} \frac{1}{\frac{g_{n'}}{g_n} e^{h\nu/kT} - \frac{B_{nn'}}{B_{n'n}}}$$

$$I_\nu = \frac{A_{nn'}}{\cancel{B_{n'n}}} \frac{\cancel{B_{n'n}}}{B_{nn'}} \frac{1}{\frac{g_{n'}}{g_n} \frac{B_{n'n}}{B_{nn'}} e^{\frac{h\nu}{kT}} - 1}$$

For a black body $\quad I_\nu = B_\nu(T)$

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$\Rightarrow \begin{cases} \dfrac{A_{nn'}}{B_{nn'}} = \dfrac{2h\nu^3}{c^2} \\ \dfrac{g_{n'}}{g_n} \dfrac{B_{n'n}}{B_{nn'}} = 1 \iff g_{n'} B_{n'n} = g_n B_{nn'} \end{cases}$$

$A_{nn'}$, $B_{nn'}$ et $B_{n'n}$ are the Einstein coefficients
If one is known, the others are also known.

High Frequencies $\boxed{\frac{h\nu}{kT} \gg 1}$

Wien law

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$$

and

$$I_\nu = \frac{A_{nn'}}{B_{nn'}} \frac{1}{\underbrace{\frac{g_{n'}}{g_n} \frac{B_{n'n}}{B_{nn'}}}_{=1} e^{h\nu/kT} - 1}$$

$$I_\nu B_{nn'} = A_{nn'} \underbrace{e^{-h\nu/kT}}_{\ll 1}$$

Spontaneous emission dominates $\Rightarrow B_{nn'} I_\nu \ll A_{nn'}$

$A_{nn'} \sim 10^8$ à $10^9\, s^{-1}$

radiative time of one atom $\sim 10^{-8}\, s$

Low frequencies. $\boxed{\frac{h\nu}{kT} \ll 1}$

$$e^{h\nu/kT} \sim 1 + \left(\frac{h\nu}{kT}\right) + \cdots$$

$$B_{nn'} I_\nu = A_{nn'} \underbrace{\frac{kT}{h\nu}}_{\gg 1} = B_{nn'} \frac{2h\nu^3}{c^2} \frac{kT}{h}$$

Stimulated emission dominates

An intuitive explanation of the stimulated emission process

For significant stimulated emission to occur, the upper energy level $n$ must be sufficiently populated relative to the lower level $n'$. Ideally a population inversion is required: $N_n \gg N_{n'}$

Si $kT \ll h\nu$, collisions will not be able to excite the atom to the upper energy level, which will therefore remain sparsely populated. The probability that a photon of frequency $\nu$ will encounter an excited atom is low.

In the infrared domain, $h\nu$ is small ($\lambda$ is large), and therefore $N_n \sim N_{n'}$. The probability of finding atoms in the excited state is thus comparable to that of finding them in the lower level $n'$. As a result, the probability of stimulated transitions is high $\longrightarrow$ MASER

We will try to express $\Gamma_n$ (rate of radiative decay)

in a clearer manner in the case of Spontaneous emission

$$K_\nu' = \frac{\pi e^2}{m_e c^2} N_{n'} f_{n'n} \frac{\Gamma_n + \Gamma_{n'}}{4\pi^2 (\nu_0 - \nu)^2 + \frac{(\Gamma_n + \Gamma_{n'})^2}{4}}$$

with $\nu_0 = \dfrac{E_n - E_{n'}}{h}$

Let's write the variation of $N_n$ with time

$$\frac{dN_n}{dt} = -N_n \sum_{n'} A_{nn'}$$

let's consider a normal atom, with a range of levels $n'' \gg 1$

$$\frac{dN_n}{dt} = -N_n \Gamma_n$$

The $e^-$ starting from $n$ can end on a range of $n'$ levels

$$\Rightarrow \quad \Gamma_n = \sum_{n'} A_{nn'} \quad \text{and} \quad N_n = N_n^0 e^{-\Gamma_n t}$$

$$\Gamma_n = \frac{1}{T_n}$$

mean lifetime of the level $n$

In the case of high temperature and high radiation density, one must consider the general case, that means take into account absorption and stimulated emission

let's consider $n' < n < n''$

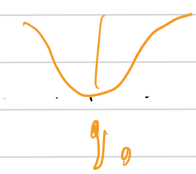
$$\Pi_n = \underbrace{\sum_{n'} A(n,n')}_{\text{spontaneous emission}} + \underbrace{\sum_{n'} B(n,n') I(\nu_{nn'})}_{\text{stimulated emission}} + \underbrace{\sum_{n''} B(n,n'') I(\nu_{nn''})}_{\text{absorption}}$$

Note : ionisation is considered in this term as absorption indeed, even if the energy level is not quantized.

Spectral lines, either in emission or absorptions, are linked to the transitions between quantized levels of an atom or molecule. These levels are not infinitely thin, hence the line is broadened.

Causes are manifold: natural width as a result of the time spent on an energy level, that can be altered by collisions, Doppler effect due to thermal motions, turbulence, large scale motions, etc

Line profile: $\phi(\nu)$, such that $\int_0^\infty \phi(\nu)\, d\nu = 1$



$\phi(\nu)\, d\nu$ = probability of absorption

$\phi(\nu)$ is centered on $\nu_0$

Variation of the population of the lower level:

$$\frac{dN_{n'}}{dt} = A_{nn'} N_n + B_{nn'} N_n \bar{u} - B_{n'n} N_{n'} \bar{u}$$

$$\bar{u} = \int_0^\infty u_\nu \phi(\nu)\, d\nu$$

Energy density of the radiation averaged over the line profile.

Spectral (in $\nu$) and volume density of radiation: $u_\nu = \frac{1}{c} \int I_\nu \, d\Omega$

$u_\nu = \frac{4\pi}{c} I_\nu$

and $$\frac{dI_\nu}{ds} = \rho j_\nu - k_\nu \rho I_\nu \; (= E + S + A)$$

$\ell \equiv$ line

One can demonstrate $$j_\nu^\ell \rho = \frac{N_n}{4\pi} A_{nn'} \phi(\nu) h\nu$$

and $$K_\nu^\ell \rho = \frac{h\nu}{4\pi} \phi(\nu) N_{n'} B_{n'n} \left[1 - \frac{B_{nn'}}{B_{n'n}} \frac{N_n}{N_{n'}}\right]$$

If the populations of the energy levels follow the Boltzmann

$$K_\nu^\ell \rho = \frac{h\nu}{4\pi} \phi(\nu) N_{n'} B_{n'n} \left[1 - \underbrace{\frac{g_n}{g_{n'}} \frac{B_{nn'}}{B_{n'n}}}_{=1} e^{-h\nu/kT}\right]$$

(black body)

Previous estimate:

$$K_\nu' = \frac{\pi e^2}{m_e c} N_{n'} f_{n'n} \frac{\Gamma_n + \Gamma_{n'}}{4\pi^2 (\nu_0 - \nu)^2 + \left(\frac{\Gamma_n + \Gamma_{n'}}{4}\right)^2}$$

was not corrected for the stimulated emission

It is now corrected

$$K_\nu' = \frac{h\nu}{4\pi} \phi(\nu) N_{n'} B_{n'n} \underbrace{\left[1 - e^{h\nu/kT}\right]}_{\text{correction factor}}$$

<u>Valid in LTE</u>

Source function

$$S_\nu^l = \frac{j_\nu^l}{k_\nu^l} = \frac{N_n A_{nn'} \phi(\nu) h\nu}{h\nu \, \phi(\nu) \left[ N_{n'} B_{n'n} - N_n B_{nn'} \right]}$$

$$S_\nu^l = \frac{N_n A_{nn'}}{N_{n'} B_{n'n} - N_n B_{nn'}}$$

we have

$$\left. \begin{aligned} \frac{B_{n'n}}{B_{nn'}} &= \frac{g_n}{g_{n'}} \\ \text{and} \quad \frac{A_{nn'}}{B_{nn'}} &= \frac{2h\nu^3}{c^2} \end{aligned} \right\} \rightarrow \quad S_\nu^l = \frac{A_{nn'}}{B_{nn'}} \frac{1}{\frac{N_{n'}}{N_n} \frac{B_{n'n}}{B_{nn'}} - 1}$$

$$S_\nu^l = \frac{2h\nu^3}{c^2} \frac{1}{\frac{N_{n'}}{N_n} \frac{g_n}{g_{n'}} - 1}$$

There are three interesting cases:

1. Thermal emission (LTE), small thermal variation
   satisfied when number of transitions per collisions $\gg$ transition by radiation

2. Non thermal emission (However with normal populations)

3. Population inversion $\quad \frac{N_{n'}}{g_{n'}} < \frac{N_n}{g_n}$

1. Thermal emission

$$\frac{N_{n'}}{N_n} = \frac{g_{n'}}{g_n} e^{\frac{h\nu}{kT}} \Rightarrow S_\nu = B_\nu(T)$$

2. Non-thermal emission

$$\frac{N_{n'}}{N_n} \neq \frac{g_{n'}}{g_n} e^{h\nu / kT}$$

Populations are not described by a Boltzmann distribution.

The population of each level must be calculated

$$\left( e^{-\frac{h\nu}{kT}} < 1 \quad \text{even beyond LTE} \right)$$

3. Population inversion

The absorption coefficient becomes negative

$$K_\nu^l \rho \propto \left( 1 - \frac{B_{nn'}}{B_{nn'}} \frac{N_n}{N_{n'}} \right) = \left( 1 - \underbrace{\frac{g_{n'}}{g_n} \frac{N_n}{N_{n'}}}_{>1} \right)$$

$$K_\nu^l \rho < 1$$

The intensity increases !!

if $\tau \sim 100$ $\quad I_\nu(h) = I_\nu(0)\, e^{-K_\nu \rho} = I_\nu(0)\, e^{-\tau_\nu}$

$$e^{100} \sim 10^{43} \; !$$

We had derived some relations involving oscillator forces

What are these forces?

let's consider the absorption only: $\rho \kappa_\nu^{l} = \frac{N_{n'}}{4\pi} B_{n'n} h\nu \, \phi(\nu)$

coefficient of absorption for one line $=$ $\alpha = \int^{line} \alpha_\nu \, d\nu$

$$\int_{line} \rho \, \kappa_\nu^{l} \, d\nu = \frac{N_{n'}}{4\pi} B_{n'n} \, h\nu \int_{line} \phi(\nu) \, d\nu$$

$\to \nu_{min} - \nu_{max}$

$$= \frac{N_{n'}}{4\pi} B_{n'n} \, h\nu \qquad \left( \int_{line} \phi(\nu) \, d\nu = 1 \right)$$

$$= \frac{\pi e^2}{m_e c} N_{n'} f .$$

$$\Rightarrow f = \frac{m_e c}{4\pi e^2} B_{n'n} \, h\nu = 7.484 \, 10^{-7} \frac{B_{n'n}}{\lambda(\text{Å})}$$

$$B_{nn'} = \frac{c^2}{2 h \nu^3} \frac{g_n}{g_{n'}} A_{nn'}$$

$$\Rightarrow f = \frac{m_e c^3}{8\pi^2 e^2 \nu^2} \frac{g_n}{g_{n'}} A_{nn'} = 1.884 \, 10^{-15} \lambda^2 \frac{g_n}{g_{n'}} A_{nn'}$$

Relation between the relation force of a line in emission / absorption

$$g_n f_{em} = g_{n'} f_{abs}.$$

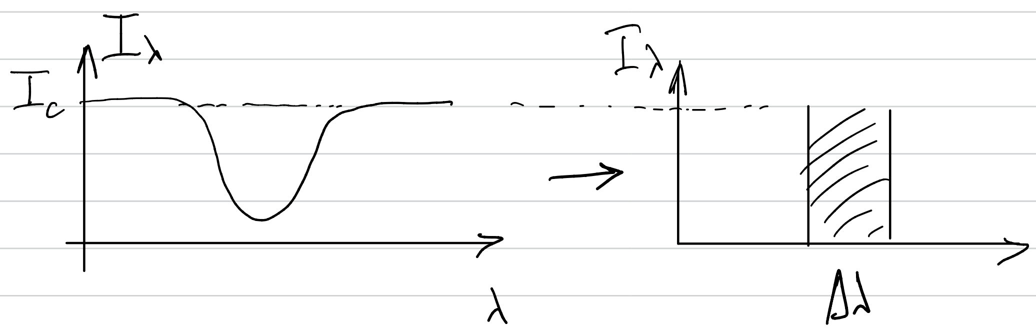
$\log gf$ is tabulated

is calculed from atomic physics

## Broadening mechanisms

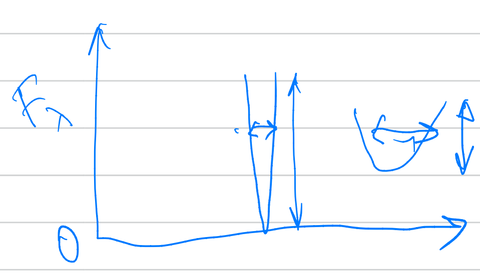
How do we measure a line?

### Equivalent width




$$W_\lambda = \int_0^\infty \frac{I_c - I_\lambda}{I_c} \, d\lambda$$

$$[W_\lambda] = \mathring{A}$$



## (A) Broadening by collisions

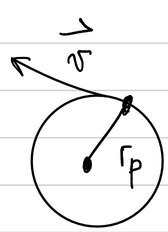
The interactions and collisions between particles result in additional de-excitation $\rightarrow$ shorten the lifetime of the upper energy level n. This induces a broadening of the lines, that involve this level. Lorentz shape broadening

There are 2 possible approaches, depending on the physical conditions

(1) One considers each collision individually, making the hypothesis that the radiative particle is not perturbed between 2 interactions and that the duration of the interaction is short compared to the time between 2 interactions.

(2) One associates a global effective potential with the interactions which can change with time

(1) To get an order of magnitude of $\gamma_{coll}$, damping constant, one considers $\tau_{coll}$ mean duration time between collisions




$$\gamma_{coll} \sim \frac{1}{\tau_{coll}} \sim n_p \pi r_p^2 v$$

$n_p$ = number of particles / volume

$v$ = particle velocity relative to the radiative atom

$r_p$ = interaction radius

$$v = \sqrt{kT\left(\frac{1}{m_p} + \frac{1}{m_a}\right)}$$ depends on the medium temperature and type of particle.

$m_a$ = atom mass

$m_p$ = mass of the perturbating particle.

(2) The interaction potential depend on the type of particle

→ If neutral hydrogen atom, dipole potential Van der Waals

Force in $\frac{1}{r^6}$

→ If electrons or ions, Coulomb potential

Force in $\frac{1}{r^2}$ (+ long range)

So, contrary to the natural width, the collisional broadening depends on the density of particles, temperature, ionisation rate of the medium. One can use the line widths to infer these parameter

$$\gamma_{tot} = \gamma_{natural} + \gamma_{coll}.$$

(damping factor in the initial equation of motion)

## (B) Thermal broadening

Each atom absorbs at frequency $\nu_0$ in its reference frame but at $\nu = \nu_0 \left(1 + \frac{v_x}{c}\right)$ in the observer reference frame because of its thermal speed

$v_x \equiv$ velocity towards observer

In 1 dimension ; $f(v_x) = \frac{dN_x}{N} = \sqrt{\frac{m}{2\pi k T}}\, e^{-\frac{m v_x^2}{2kT}} dv_x$

In LTE the velocity distribution follows a Maxwell distribution

$\frac{dN_x}{N}$ = distribution function of the particles that can intercept photons of frequence $\nu$, resulting from their velocity shift.

$$\nu = \nu_0 \left(1 + \frac{v_x}{c}\right) \Rightarrow v_x = \frac{c}{\nu_0}(\nu - \nu_0) \quad dv_x = \frac{c}{\nu_0} d\nu$$

$$\frac{dN_x}{N} = \sqrt{\frac{m}{2\pi k T}}\, \frac{c}{\nu_0} \exp\left[-\frac{mc^2}{2kT\nu_0^2}(\nu - \nu_0)^2\right] d\nu$$

Doppler width : $\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$

$$f(v_x) = \frac{dN_x}{N} = \frac{1}{\sqrt{\pi}\,\Delta\nu_D} \exp\left(-\frac{(\nu - \nu_0)^2}{\Delta\nu_D^2}\right) d\nu$$

$f(v_x)$ is a gaussian with full width at half maximum

$$\Delta\lambda_{1/2} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT\ln 2}{m}} \equiv 2\sqrt{\ln 2}\,\Delta\nu_D$$

One must sometimes take into account global motions, such as turbulence in the gas, in addition to thermal motion. In such a case, one can add $v_{turb}$ to the thermal velocity and the Doppler broadening is written as:

$$\Delta\nu_D^{eff} = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m} + v_{turb}^2}$$

Note: $\phi(\nu) = \dfrac{dN_x}{N} = \dfrac{I_\nu\, d\nu}{I}$ avec $I = \displaystyle\int_{raie} I_\nu\, d\nu$

if each atom absorbs or emits an infinitely thin line.

## (C) Combining Doppler, natural, and collisional profiles

Each of these processes leads to line broadening. Combining them results in a profile which is the convolution of each, making the assumption that they are independent.

Gaussian profile [Thermal Doppler and turbulence] $*$ Lorentz profiles [Natural and Collisions]
= Voigt profile

## Voigt function

$$H(a,u) = \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{a^2 + (u-y)^2} \, dy$$

with $a = \frac{\gamma}{4\pi \Delta\nu_D}$ $\quad u = \frac{\nu - \nu_0}{\Delta\nu_D}$ $\quad y = \frac{\nu'}{\Delta\nu_D}$

There are other sources of broadening

→ Macroscopic Doppler effect (macroturbulence, rotation)
→ Zeeman effect (magnetic field)
→ Stark effect (electric field, for H and He)

Doppler width of a line at 5000 Å due to thermal motion

$$\Delta\nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$$

One must choose an element here Fe

$$\Delta\lambda_D = \frac{c}{\nu_0^2} \Delta\nu_D = \frac{\lambda_0^2}{c} \Delta\nu_D$$

$$\Delta\lambda_D = \frac{\lambda_0}{c} \sqrt{\frac{2kT}{56 \, m_H}}$$

$k = 1.381 \; 10^{-23} \; J.K^{-1}$
$c = 2.99 \; 10^{8} \; m.s^{-1}$
$m_H = 1.63 \; 10^{-27} \; kg.$

$$\Delta\lambda_D \approx 0.022 \, \mathring{A}$$

How does the width of a line depend on the mass of the chemical element?

$$\Delta\nu_D \propto \frac{1}{\sqrt{m}} \Rightarrow$$ The more massive the element the narrower the line

Continuous absorptions.

Corresponds to electrons scattering

Let's consider again the classical equations with $\hat{n} = n - in'$

$$\begin{cases} n - 1 = \dfrac{Ne^2}{2\pi m_e} \dfrac{\nu_0^2 - \nu^2}{(\nu_0^2 - \nu^2)^2 + \dfrac{\gamma^2\nu^2}{4\pi^2}} \\ K'_\nu = \dfrac{\pi N e^2}{m_e c} \dfrac{4\gamma\nu^2}{4\pi^2(\nu_0^2 - \nu^2)^2 + \gamma\nu^2} \end{cases}$$

Unbound case: We are far from resonance $\nu \gg \nu_0$

the terms in $\nu_0^2 - \nu^2$ dominate

$$n - 1 = \frac{Ne^2}{2\pi m_e} \frac{1}{\nu_0^2 - \nu^2} \; ; \; \gamma = \frac{2}{3} \frac{e^2\omega^2}{m_e c^3} = \frac{8\pi^2 e^2 \nu^2}{3 m_e c^3} \; ; \; K'_\nu = \frac{8\pi N e^4}{3 m_e e^4} \frac{1}{\left[\left(\frac{\nu_0}{\nu}\right)^2 - 1\right]^2}$$

$\nu \gg \nu_0$ $\quad K'_\nu$ can be further simplified $\quad , \quad K'_\nu = K_\nu \rho = \frac{8\pi N e^4}{3 m_e^2 c^4} = cste$

Thomson opacity

For free electrons $\quad K_\nu \rho = 0.66\ 10^{-24}\ n_e\ cm^{-1}$

isotropic and independent of $\nu$

- The Thomson cross-section $\sigma_{Th}$ is defined as $K_\nu \rho = n_e \sigma_{Th}$

$$\sigma_{Th} = \frac{K_\nu \rho}{n_e} = 0.66\ 10^{-24}\ cm^2.$$

$$n_e = \frac{\rho}{2 m_H}(1+X) \quad ; \quad X \equiv \text{Hydrogen mass fraction}.$$

- Opacity of Thomson scattering $\quad K = \frac{K'}{\rho} = \frac{\sigma_{Th}\, n_e}{\rho}$

$$K = \frac{\sigma_{Th}}{2 m_H}(1+X) \simeq 0.2\,(1+X)\ cm^2 \cdot g^{-1}$$

Given $\sigma_{Th} \propto \frac{1}{m^2}$ $\quad$ Diffusion by the nuclei is negligible.

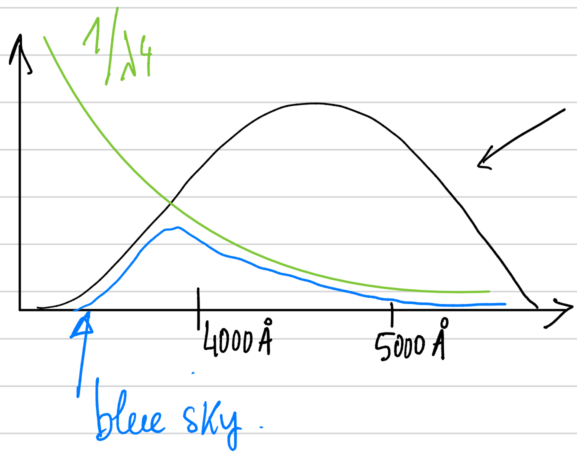
$$\left[ \sigma_{Th} = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{mc^2}\right)^2 \equiv \left(\frac{\text{charge}^2}{\text{rest mass}^2}\right)^2 \right]$$

$\rightarrow$ important factor in the atmosphere of the hottest stars.

Bound case $\nu_0 \gg \nu$

$$K_{\nu}' = \frac{8\pi N e^4}{3 m_e^2 c^4} \frac{1}{\left[\left(\frac{\nu_0}{\nu}\right)^2 - 1\right]^2} \sim \frac{8\pi N e^4}{3 m_e c^4} \frac{\nu^4}{\nu_0^4} \propto \frac{1}{\lambda^4}$$

Rayleigh scattering




Blue is more scattered than the rest

The Rayleigh scattering can be important in the atmosphere of G and K-type stars [5000K – 6000K, 3500 – 5000K, respectively] in wich most of the atoms are H in its fundamental state. visible photons have $\nu \ll \nu_0$ where $\nu_0$ is the lyman series frequency.

"Towards the curve of growth"

let's consider the transfer equation in its global form

$$\frac{dI_\nu}{ds} = \rho j_\nu - \kappa_\nu \rho I_\nu$$

optical depth : $\tau_\nu = \int \kappa'_\nu \, ds \quad \equiv \quad d\tau_\nu = \kappa'_\nu ds$

$$\Rightarrow \quad \frac{dI_\nu}{ds} = S_\nu - I_\nu$$

let's consider the case of an absorption line, no emissivity and homogeneous medium

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu$$

$$\Leftrightarrow \quad \frac{dI_\nu}{d\tau_\nu} = -I_\nu$$

$$\Rightarrow \quad I_\nu = I_{\nu,c} \, e^{-\tau_\nu} \quad \text{with } I_{\nu,c} \text{ continuum}$$

$\tau_\nu$ varies with $\nu$ in the line

One can normalize the spectrum $\mathcal{I}_\nu = \dfrac{I_\nu}{I_{\nu,c}} = e^{-\tau_\nu}$

similarly in $\lambda$ : $\mathcal{I}_\lambda = \dfrac{I_\lambda}{I_{\lambda,c}} = e^{-\tau_\lambda}$

Equivalent width: (EQW)

$$W_\lambda = \int_0^\infty \frac{I_{\lambda,c} - I_\lambda}{I_{\lambda,c}}$$

$$W_\nu = \int_0^\infty \frac{I_{\nu,c} - I_\nu}{I_{\nu,c}}$$

$$W_\lambda = \left(\frac{\lambda_0}{c}\right)^2 W_\nu$$

For an unresolved source: $W_\lambda = \int_0^\infty \frac{F_{\lambda,c} - F_\lambda}{F_{\lambda,c}} d\lambda$

The equivalent width is a direct estimate of the number of absorbing atoms on the line of sight

Property of the EQW: It is not modified by the spectrograph.

Indeed, the convolution by a normalized profile preserves the integral.

if $h = g * \phi$ $\qquad \phi$ normalized profile

$$\int h(x)\,dx = \int g(x)\,dx \int \phi\,dx = \int g(x)\,dx$$

For a Voigt profile $W_\nu = \int \left[1 - e^{\tau_{\nu,0} H(a,u)}\right] d\nu$

with $\tau_\nu = \tau_{\nu,0} H(a,u)$

let's be more explicit now; let's consider a layer of thickness $\ell$ ; N the concentration of the population of interest , f the oscillator strength.

$$\tau_\nu = \kappa_\nu^{\ell'} \ell \qquad \text{(homogeneous medium)}$$

$$= a_\nu N \ell$$

$$= a_0 N \ell \, \phi_\nu$$

with $$a_0 = \frac{\pi e^2}{m_e c} f \left( 1 - e^{-\frac{h\nu}{kT}} \right)$$

Indeed we had demonstrated : $$\kappa_\nu' = \frac{h\nu}{4\pi} \phi(\nu) N_{n'} B_{n'n} \left[ 1 - e^{-h\nu/kT} \right]$$

and : $$f = \frac{m_e c}{4\pi^2 e^2} h\nu B_{n'n}$$

$$\rightarrow \kappa_\nu' = f \phi(\nu) N_{n'} \frac{\pi e^2}{m_e c} \left[ 1 - e^{-h\nu/kT} \right]$$

$$\Rightarrow W_\nu = \int_0^\infty \left( 1 - e^{-N \ell a_0 \phi_\nu} \right) d\nu$$

## Weak lines

optically thin medium: $\tau_\nu \ll 1$

$$\Rightarrow N l a_0 \phi_\nu \ll 1$$

partial fraction decomposition $\longrightarrow$ $W_\nu = N l a_0 \int_0^\infty \phi_\nu \, d\nu = N l a_0$

Curve of growth: variation of $W$ as a function of the abundance of a chemical element. It is linearly proportional to $N$ and $Nl \equiv$ projected density $\equiv$ column density.

Validity: weak spectral lines, interstellar lines in the optical and radio domains

Very favorable domain to derive abundances.

## Strong lines

When the number of absorbers increase, the center of the line becomes increasingly optically thick. The contribution of the line wings is still negligible, however even if $N$ increases, the absorption does not vary much

One can show that $W$ varies as $\sqrt{\log N}$

The absorption is still due to the Doppler effect.

## Very strong lines

The wings of the lines cannot be neglected anymore

$W$ varies as $\sqrt{N}$